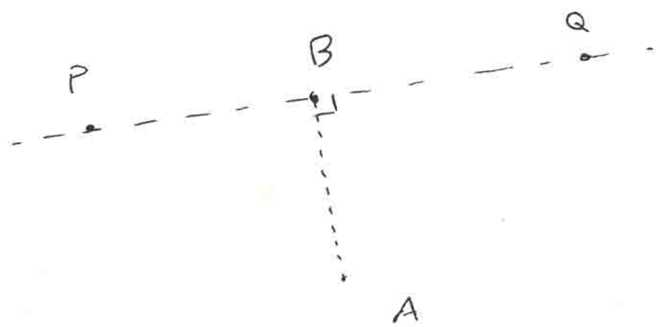
Solutions to Practice Exams:

Fall 2001

1. a) $P = (2,-2,4)$, $Q = (3,-4,6)$, $A = (1,2,0)$

$$\vec{PQ} = \langle 3-2, -4-(-2), 6-4 \rangle = \langle 1, -2, 2 \rangle$$

$$\vec{PA} = \langle 1-2, 2-(-2), 0-4 \rangle = \langle -1, 4, -4 \rangle$$



$$\vec{PB} = \text{proj}_{\vec{PQ}} \vec{PA} = \frac{\vec{PQ} \cdot \vec{PA}}{|\vec{PQ}|^2} \vec{PQ}$$

$$= \frac{\langle 1,-2,2 \rangle \cdot \langle -1,4,-4 \rangle}{|\langle 1,-2,2 \rangle|} \langle 1,-2,2 \rangle$$

$$= \frac{-1-8-8}{\sqrt{1+4+4}} \langle 1,-2,2 \rangle$$

$$= -\frac{17}{3} \langle 1,-2,2 \rangle = \left\langle \frac{-17}{3}, \frac{34}{3}, -\frac{34}{3} \right\rangle$$

b) $P = (2,-2,4)$ $\quad \vec{PB} = \left\langle -\frac{17}{3}, \frac{34}{3}, -\frac{34}{3} \right\rangle$

$B = (a, b, c)$

$a - 2 = \frac{-17}{3} \qquad b + 2 = \frac{34}{3} \qquad c - 4 = \frac{-34}{3}$

$a = \frac{-17}{3} + 2 \qquad b = \frac{34}{3} - 2 \qquad c = \frac{-34}{3} + 4$

$a = -\frac{11}{3} \qquad b = \frac{28}{3} \qquad c = -\frac{22}{3}$

$$B = \left(-\frac{11}{3}, \frac{28}{3}, -\frac{22}{3}\right)$$

c) $\vec{AB} = \left\langle -\frac{11}{3} - 1, \frac{28}{3} - 2, -\frac{22}{3} - 0 \right\rangle$

$= \left\langle -\frac{14}{3}, \frac{22}{3}, -\frac{22}{3} \right\rangle$

Distance is $|\vec{AB}| = \sqrt{\left(-\frac{14}{3}\right)^2 + \left(\frac{22}{3}\right)^2 + \left(\frac{22}{3}\right)^2}$

$= \sqrt{\frac{1164}{9}}$

$= \sqrt{\frac{388}{3}}$

2. a) If A is on both lines there are numbers s and t where

$$1+3t=-1-s \qquad 3-3t=1+5s \qquad 2-4t=8-2s$$

or $3t=-2-s \qquad 3t=2-5s$

or $-2-s=2-5s$

$$4s=4$$

$$s=1$$

$$t=-1$$

Note: These work in the third equation!

So A is $(-2,6,6)$

b) Take $t=0$, $B=(1,3,2)$.

c) The direction vectors of the lines are $\vec{v}_1=\langle 3,-3,-4\rangle$, $\vec{v}_2=\langle -1,5,-2\rangle$. These are both in the desired plane, so we can take as our normal vector

$$\vec{n}=\vec{v}_1\times\vec{v}_2=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -3 & -4 \\ -1 & 5 & -2 \end{vmatrix}=\begin{vmatrix} -3 & -4 \\ 5 & 2 \end{vmatrix}\vec{i}-\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix}\vec{j}+\begin{vmatrix} 3 & -3 \\ -1 & 5 \end{vmatrix}\vec{k}$$

$$=(-6+20)\vec{i}-(-6-4)\vec{j}+(15-3)\vec{k}$$

$$=\langle -14, 10, 12\rangle$$

and use A as our point. The plane is

$$-14(x+2)+10(y-6)+12(z-6)=0$$

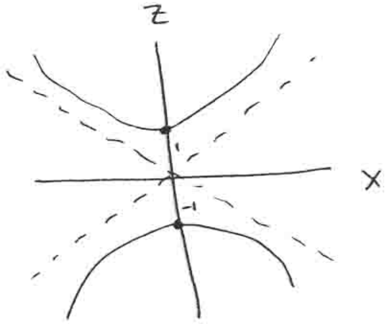
or $-14x+10y+12z-160=0$ or $-7x+5y+6z-80=0$

3. a) on XZ plane $y=0$, so we have

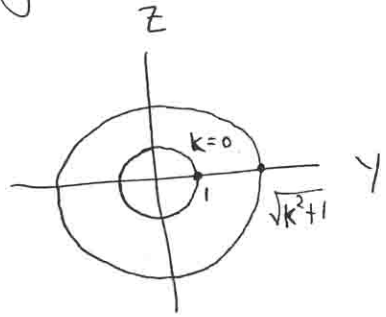
$$x^2+1=z^2$$

$$z^2-x^2=1$$

hyperbola

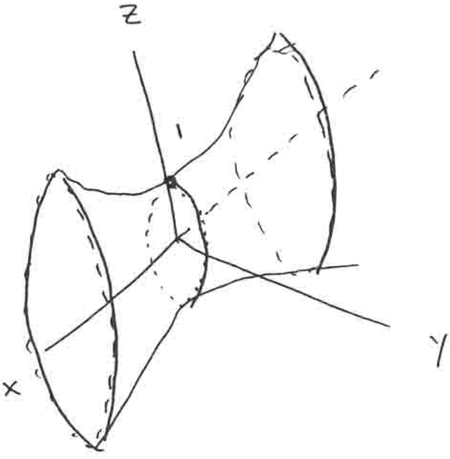


b) We have $k^2+1=y^2+z^2$, circles centered at the origin of radii $\sqrt{k^2+1}$



The smallest circle is the unit circle for $k=0$, and they get larger as k gets larger (or smaller!).

c) Note for the XY plane we get hyperbolas as well (just like part a)).

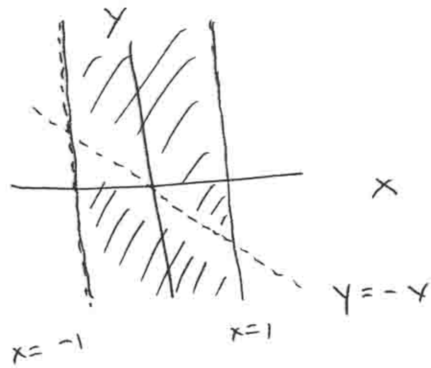


Hyperboloid of one sheet.

1. a) we need $1-x^2 \geq 0$ and $x+y \neq 0$, so

$x^2 \leq 1$ (i.e. $-1 \leq x \leq 1$) and $x+y \neq 0$

or $x \neq -y$



It's the stripe without the dotted line!

b) Suppose we let $(x,y) \to (0,0)$ along the line $x=0$, then

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{2x^2y^2}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{0}{y^4} = 0$$

But also let $(x,y) \to (0,0)$ along the line $y=x$, then

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{2x^2y^2}{x^4+y^4} = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{2x^4}{2x^4} = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} 1 = 1$$

So the limit cannot exist.

5. a) $\vec{v}(t) = \langle t, \sqrt{2}\, t^{1/2}, 1\rangle$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \tfrac{1}{2}t^2, \sqrt{2}\cdot\tfrac{2}{3}t^{3/2}, t\rangle + \vec{c}$$

but $\vec{r}(0) = \vec{c} = \langle 0, -\tfrac{2}{3}, 0\rangle$

So $\vec{r}(t) = \left\langle \tfrac{1}{2}t^2, \tfrac{2\sqrt{2}}{3}t^{3/2} - \tfrac{2}{3}, t\right\rangle$

b) If it passes through $(2,2,2)$ at that point $t=2$ (from z-coordinate), and this makes sense since $\tfrac{1}{2}(2)^2 = 2$ and

$$\frac{2\sqrt{2}}{3}2^{3/2} - \frac{2}{3} = \frac{2\sqrt{2}\cdot 2\sqrt{2}}{3} - \frac{2}{3} = \frac{8}{3} - \frac{2}{3} = \frac{6}{3} = 2$$

c) Since the z-coordinate is always increasing on this time interval, we trace out the curve exactly once, and

$$L = \int_0^1 |\vec{r}\,'(t)|\,dt = \int_0^1 |\vec{v}(t)|\,dt$$

$$= \int_0^1 |\langle t, \sqrt{2}t^{1/2}, 1\rangle|\,dt$$

$$= \int_0^1 \sqrt{t^2+2t+1}\,dt$$

$$= \int_0^1 \sqrt{(t+1)^2}\,dt$$

$$= \int_0^1 t+1\,dt$$

$$= \left(\tfrac{1}{2}t^2 + t\right)\Big|_0^1$$

$$= \tfrac{1}{2} + 1$$

$$= \tfrac{3}{2}$$

Note: $(t+1) > 0$ on $[0,1]$