Fall 2002

1. a) IGNORE

b) The normal vectors are

$\vec{n}_1 = \langle 2, -3, 4 \rangle \qquad \vec{n}_2 = \langle 0, 1, -1 \rangle$

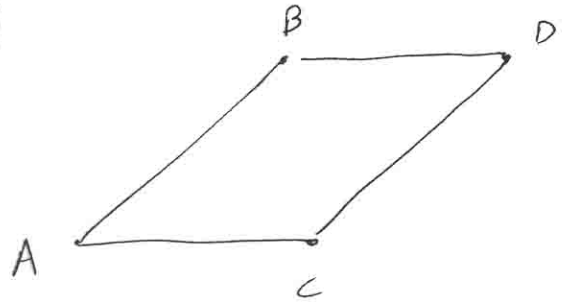
So the angle is

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{-3-4}{\sqrt{4+9+16}\sqrt{1+1}} = \frac{-7}{\sqrt{58}}$$

whatever θ turns out to be...

$\theta = 2.736\ldots$ rads

c)



$A = (1, 5, 3)$

$B = (4, 1, 5)$

$D = (3, 2, -5)$

<u>Note</u> : $\vec{AC} = \vec{BD}$ $\quad \vec{AB} = \vec{CD}$

So if $C = (a, b, c)$

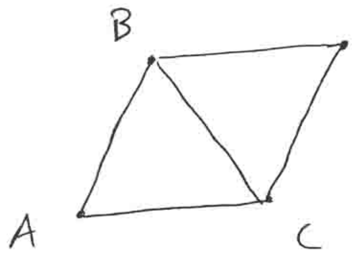
$\langle a-1, b-5, c-3 \rangle = \langle -1, 1, -10 \rangle$

$\langle 3-a, 2-b, -5-c \rangle = \langle 3, -4, +2 \rangle$

So $a = 0 \quad b = 6 \quad c = -7$

$C = (0, 6, -7)$

2. a)



Area of triangle is $\frac{1}{2}$ area of parallelogram

So Area $= \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right| = \frac{1}{2} \left| \langle -1, 1, 0 \rangle \times \langle -1, 0, 1 \rangle \right|$

$= \frac{1}{2} \left| \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \right| = \frac{1}{2} \left| \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} \vec{k} \right|$

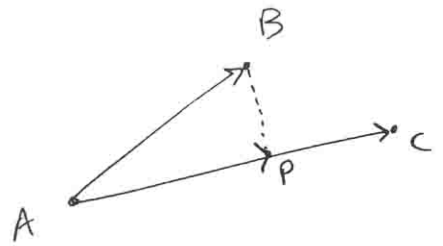
$= \frac{1}{2} \left| \langle 1, 1, 1 \rangle \right| = \frac{1}{2} \sqrt{1+1+1} = \frac{\sqrt{3}}{2}$

b) $$\text{proj}_{\vec{AB}}(\vec{AC}) = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|^2}\,\vec{AB}$$

$$= \frac{\langle -1,1,0\rangle \cdot \langle -1,0,1\rangle}{|\langle -1,1,0\rangle|^2}\cdot \langle -1,1,0\rangle$$

$$= \frac{1+0+0}{1+1}\cdot\langle -1,1,0\rangle$$

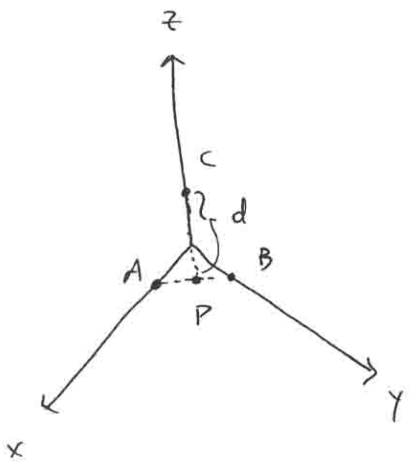
$$= \langle -\tfrac{1}{2}, \tfrac{1}{2}, 0\rangle$$



$\vec{AP} = \langle -\frac{1}{2}, \frac{1}{2}, 0\rangle$ or $\langle p_1 - 1, p_2, p_3\rangle = \langle -\frac{1}{2}, \frac{1}{2}, 0\rangle$

So $P = (\frac{1}{2}, \frac{1}{2}, 0)$

c)



$$d = |\vec{PC}| = |\langle \tfrac{1}{2}, \tfrac{1}{2}, -1\rangle|$$

$$= \sqrt{\tfrac{1}{4} + \tfrac{1}{4} + 1}$$

$$= \frac{\sqrt{6}}{2}$$

and this is the right distance since $\vec{AB}\cdot\vec{PC} = 0$.

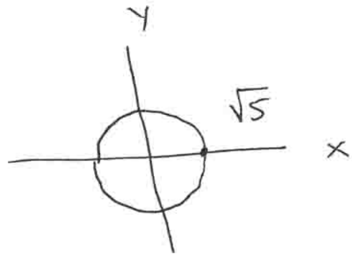
d) Note : $d|\vec{AB}|$ is the area of the triangle multiplied by 2, so they are equal!

$$2a = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$d|\vec{AB}| = \frac{\sqrt{6}}{2} \cdot |\langle -1,1,0\rangle| = \frac{\sqrt{6}}{2} \cdot \sqrt{2} = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$$

3. a) $z=3$: $9 = x^2+y^2+4$

$$x^2+y^2=5$$



Circle centered at the origin

In the xy-plane, $z=0$, and we get

$$x^2+y^2+4=0$$

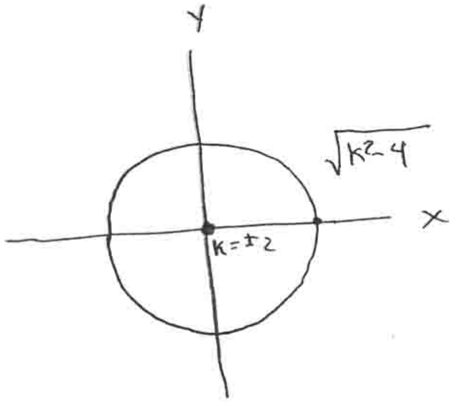
There is no trace (no solutions to this equation)!

b) $z=k$: $\quad x^2+y^2 = k^2-4$

clearly $\quad k^2-4 \geq 0$

$k^2 \geq 2$

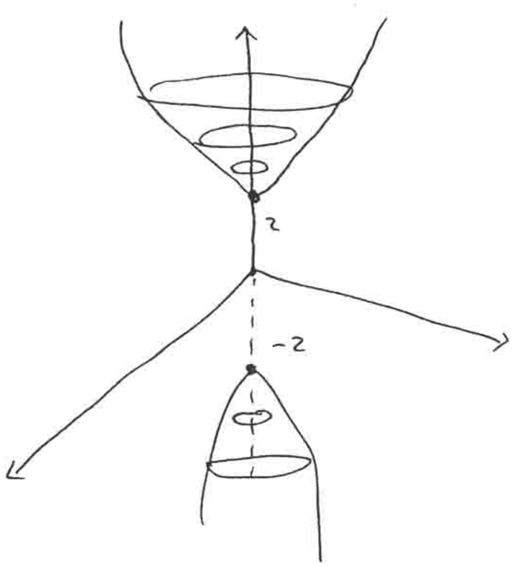
$k \geq 2$ or $k \leq -2$ $\quad$ or no trace



circles centered at the origin

Same traces for $\pm k$

c) For $x=k$, $y=k$ we get hyperboloids, so this is a hyperboloid of 2 sheets.



4. L_1 is $\dfrac{x-4}{1} = \dfrac{y-3}{1} = \dfrac{z-1}{2}$

L_2 is $\dfrac{x-1}{-1} = \dfrac{y}{2} = \dfrac{z-1}{1}$

a) Direction vecters for L_1, L_2 are

$\vec{v}_1 = \langle 1, 1, 2 \rangle$

$\vec{v}_2 = \langle -1, 2, 1 \rangle$

Suppose $\vec{v}$ is a direction vector for L,

Since $\vec{v} \cdot \vec{v}_1 = \vec{v} \cdot \vec{v}_2 = 0$ we can take

$$\vec{v} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} \vec{k}$$

$$= \langle -3, -3, 3 \rangle$$

So the line is
$$x = 3 - 3t$$
$$y = 2 - 3t$$
$$z = -1 + 3t$$

b) Since $\frac{3-4}{1} = -1$

$$\frac{2-3}{1} = -1$$

and $\frac{-1-1}{2} = -1$

P is on L_1

We want t such that $(3-3t, 2-3t, -1+3t)$ is on line L_2,

$$\frac{3-3t-1}{-1} = \frac{2-3t}{2} = \frac{-1+3t-1}{1}$$

or $3t-2 = 1-\frac{3}{2}t$

$$\frac{9}{2}t = 3$$

$$t = \frac{6}{9} = \frac{2}{3}$$

So the point is $(1, 0, 1)$

c) The line segment that represents the distance (smallest distance) between the two lines itself lies on a line whose direction vector is $\vec{v}_1 \times \vec{v}_2$. Therefore, the direction vector of L (part a)) is the direction of this line. L intersects L_1 and L_2 at P and Q.

Therefore, $|\vec{PQ}| = |\langle -2, -2, 2 \rangle| = \sqrt{4+4+4}$

$= \sqrt{12} = 2\sqrt{3}$ is the distance.

5. a) $\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle -2t, 4, 0 \rangle\,dt = \langle -t^2, 4t, 0 \rangle + \vec{C}$

but $\vec{v}(0) = \vec{0} + \vec{C} = \langle -3, 0, 4 \rangle$

So $\vec{v}(t) = \langle -t^2 - 3, 4t, 4 \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle -\frac{1}{3}t^3 - 3t, 2t^2, 4t \rangle + \vec{D}$

but $\vec{r}(3) = \langle -\frac{1}{3} \cdot 27 - 9, 18, 12 \rangle + \vec{D} = \langle 0, 1, 0 \rangle$

$\Rightarrow \langle -18, 18, 12 \rangle + \vec{D} = \langle 0, 1, 0 \rangle$

So $\vec{D} = \langle 18, -17, -12 \rangle$

So $\vec{r}(t) = \langle -\frac{1}{3}t^3 - 3t + 18, 2t^2 - 17, 4t - 12 \rangle$

b) at $(0,1,0)$ $t = 3$

So ~~[illegible]~~ $\vec{v}(3) = \langle -12, 12, 4 \rangle$ is a direction vector, so the line is

$x = -12t \quad y = 1 + 12t \quad z = 4t$

c) $L = \int_0^2 |\vec{v}(t)|\,dt = \int_0^2 \sqrt{(-t^2-3)^2 + 16t^2 + 16}\,dt$

$= \int_0^2 \sqrt{t^4 + 6t^2 + 9 + 16t^2 + 16}\,dt$

$= \int_0^2 \sqrt{t^4 + 22t^2 + 25}\,dt$

There is a problem here with this integral, you do <u>not</u> need to know how to solve this!