

An Analytical Solution for a Class of Oscillators, and Its Application to Filter Tuning

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Abstract—We present a completely analytical solution to a filter-comparator oscillator system, and verify it by macromodel simulations and experiment. We discuss the applications of this kind of oscillator in a vector-locked loop system for continuous time filter tuning. We also apply our solution to the operation of a resonant switched mode inverter.

Index Terms—Automatic tuning, filter tuning, oscillator, VCO.

I. INTRODUCTION

SINE-WAVE oscillators (Fig. 1) contain an active element with sufficient power gain at the oscillation frequency, a frequency selective network, and an amplitude stabilizing mechanism. They are capable of producing a near-sinusoidal signal with good phase noise and high spectral purity.

In a sine-wave oscillator, positive feedback is used around a frequency selective circuit to drive the poles of the corresponding closed-loop linear system into the right-half s-plane. In the case to be considered in this paper, the “gain” of the amplifier is set to ∞ , as shown in Fig. 2. Such systems are encountered in nonlinear control systems literature [1]–[3] and have been used by designers [4], [5] in filter tuning schemes, where one approach is to construct an oscillator with filter building blocks (integrators), for the purposes of monitoring and tuning filter characteristics. It is important in such schemes to make sure that the filter undergoes no internal limiting phenomena, so that its response can be predicted by linear system theory. This is in contrast to other oscillator methods, in which limiting within the filter can modify the frequency of oscillation [6] of the closed-loop system, which then does not match and track the locations of the filter poles with variations in temperature and other environmental factors.

The system of Fig. 2 has been studied earlier in the context of integrated oscillators using digital blocks. For an analysis of the system using nonlinear differential equations, the reader is referred to [7], where the comparator is realized by using a cascade of two inverters, and the bandpass filter is an LCR series circuit. The analysis in the above work is done by approximating the nonlinear transfer characteristic of the comparator by a suitable transcendental function, and solving the nonlinear differential equation obtained using well known techniques.

Manuscript received February 10, 1997; revised October 8, 1997. This paper was recommended by Associate Editor W. Mathis.

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Publisher Item Identifier S 1057-7122(98)02532-X.

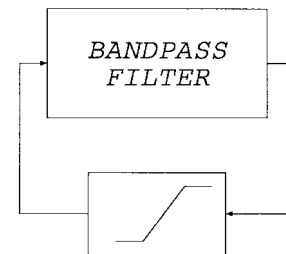


Fig. 1. Block diagram of an oscillator.

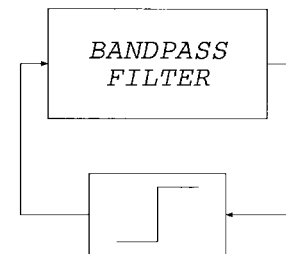


Fig. 2. The filter comparator oscillator.

The filter-comparator system could also be analyzed by using the describing function approach [2], where the nonlinear block is replaced by an “equivalent” linear block. A first-order describing function analysis, however, predicts that the system will oscillate at the filter pole frequency, regardless of the filter quality factor, which we will see is incorrect. A higher order describing function analysis gets close to the exact result. An exact method for systems consisting of linear networks and relays has been proposed by Tsytkin and is described in detail in [2]. This method, however, requires the evaluation of an infinite series using contour integrals.

The disadvantage of all the above methods for this particular system is their complexity. They do not offer much insight into system operation, and the solutions are in terms of Fourier coefficients for the steady-state response. In contrast, the solution we present in this paper is straightforward, provides intuition, and gives information about all the quantities of interest (amplitude, frequency, steady state pulse shape, build-up transient) *exactly*.

In Section II, we present our method of analysis, using the transient response, and examine the issue of Total Harmonic Distortion (THD) of the output waveform in considerable detail. Upper bounds for THD are derived for some related topologies. Section III applies the results of our analysis in the automatic tuning of continuous time filters. A new vector lock loop approach to filter tuning is presented, along with some

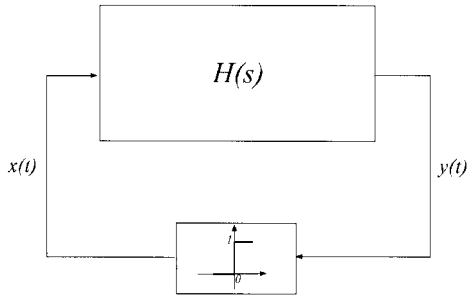


Fig. 3. Block diagram of the oscillator.

experimental results. Section IV discusses the applications of the presented analysis to switching mode resonant inverters. Section V contains the conclusions of this work.

II. OSCILLATOR TRANSIENT AND STEADY STATE

The system we analyze is shown in Fig. 3. For simplicity, until further notice we assume that the comparator output levels are 0 and 1. The filter is of the second-order bandpass type. Its transfer function is

$$H(s) = \frac{\frac{s}{\omega_0}}{\frac{s^2}{\omega_0^2} + \frac{s}{\omega_0 Q} + 1}. \quad (1)$$

In the technique to be proposed below, we will employ the step response of the filter, $s(t)$, which is

$$s(t) = \mathcal{F}^{-1} \left[\frac{H(s)}{s} \right] = \mathcal{F}^{-1} \left[\frac{1}{\omega_0} \frac{1}{\frac{s^2}{\omega_0^2} + \frac{s}{\omega_0 Q} + 1} \right] \quad (2)$$

or

$$s(t) = \frac{1}{\sqrt{1 - \frac{1}{4Q^2}}} \exp \left(\frac{-\omega_0 t}{2Q} \right) \sin \left(\omega_0 \sqrt{1 - \frac{1}{4Q^2}} t \right) u(t) \quad (3)$$

where $u(t)$ is the unit step function. The step response crosses zero whenever

$$\sin \left(\omega_0 \sqrt{1 - \frac{1}{4Q^2}} t \right) = 0 \quad (4)$$

or at times

$$t_n = nt_1, \quad n = 0, 1, 2, \dots \quad (5)$$

where

$$t_1 = \frac{\pi}{\omega_0 \sqrt{1 - \frac{1}{4Q^2}}}. \quad (6)$$

The mechanism of oscillation buildup will be described with the aid of Fig. 4. Let us assume that the system is initially relaxed, and that oscillation is triggered by a small positive noise at the comparator input at time $t = t_0 = 0$. This will cause a step input $u(t)$ to the bandpass filter. The output $y(t)$ of

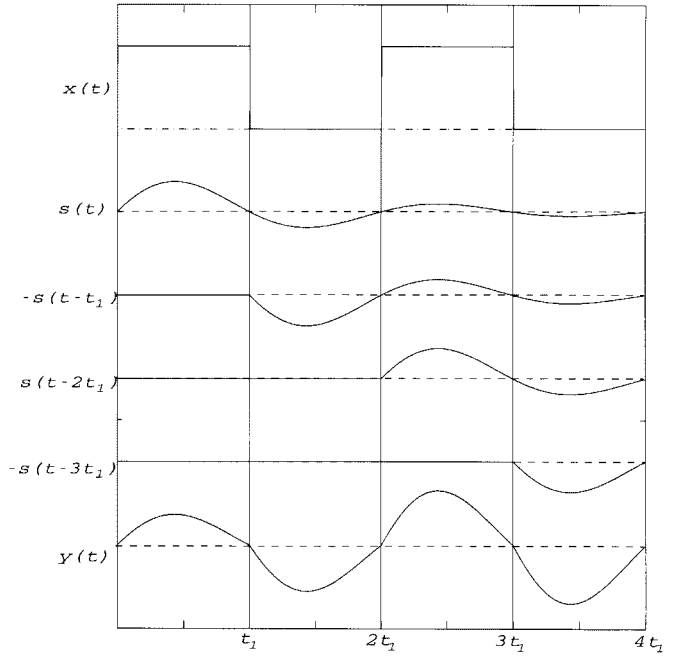


Fig. 4. Oscillation buildup mechanism.

the bandpass filter for $0 < t < t_1$ will coincide with the filter step response $s(t)$, as shown in Fig. 4. This waveform crosses zero at $t = t_1$, so at that instant the comparator switches again. Between this switching instant and the next one, the comparator output can be represented by the superposition of two steps—the first at t_0 and the second at t_1 :

$$x(t) = u(t) - u(t - t_1). \quad (7)$$

Thus, for the same interval, the output of the linear filter can be obtained using superposition as

$$y(t) = s(t) - s(t - t_1). \quad (8)$$

Notice that the zero crossings of $s(t - t_1)$ are t_1 apart from each other, just as was the case with $s(t)$. Also, the time at which $s(t - t_1)$ starts coincides with the zero crossing t_1 of $s(t)$. Thus, the output $y(t) = s(t) - s(t - t_1)$ will reach its next zero crossing when both $s(t)$ and $s(t - t_1)$ cross zero, i.e., at $t = 2t_1$. At this point, the comparator switches again, and so until the next zero crossing, its input will be

$$x(t) = u(t) - u(t - t_1) + u(t - 2t_1) \quad (9)$$

and its output will be

$$y(t) = s(t) - s(t - t_1) + s(t - 2t_1). \quad (10)$$

Reasoning as above, we conclude that the next zero-crossing will occur at $t = 3t_1$, and so on. It now becomes obvious that the output of the comparator can be represented for all positive time by

$$x(t) = \sum_{n=0}^{\infty} (-1)^n u(t - nt_1), \quad t > 0. \quad (11)$$

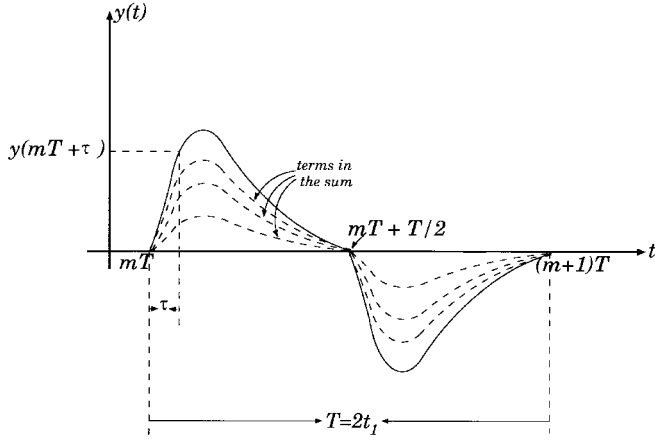


Fig. 5. Timing detail.

The filter output then is

$$y(t) = \sum_{n=0}^{\lfloor t/t_1 \rfloor} (-1)^n s(t - nt_1), \quad t > 0 \quad (12)$$

where $\lfloor t/t_1 \rfloor$ denotes the integer part of t/t_1 . It is apparent from Fig. 4 that the terms in the sum that produces $y(t)$ are positive for $nt_1 < t < (n+1)t_1$ if n is even, and negative if n is odd. By writing

$$T \equiv 2t_1 \quad (13)$$

we see that the terms in the sum are all positive for $mT < t < mT + T/2$, and negative for $mT + T/2 < t < (m+1)T$, where m is an integer. This is shown in Fig. 5.

A. Steady-State Response

The steady-state response can be obtained by using (3) in (12) and allowing t to increase. The result of this process, as shown in the Appendix, is

$$\begin{aligned} y_{ss}(mT + \tau) &= A \exp\left(-\frac{\omega_0 \tau}{2Q}\right) \sin(\omega_{osc} \tau), \\ 0 < \tau < \frac{T}{2} \\ &= A \exp\left[-\frac{\omega_0 \left(\tau - \frac{T}{2}\right)}{2Q}\right] \sin(\omega_{osc} \tau), \\ \frac{T}{2} < \tau < T \end{aligned} \quad (14)$$

where

$$\omega_{osc} = \omega_0 \sqrt{1 - \frac{1}{4Q^2}} \quad (15)$$

$$T = \frac{2\pi}{\omega_{osc}} \quad (16)$$

$$A = \frac{1}{\left[1 - \exp\left(-\frac{\pi}{\sqrt{4Q^2 - 1}}\right)\right] \sqrt{1 - \frac{1}{4Q^2}}} \quad (17)$$

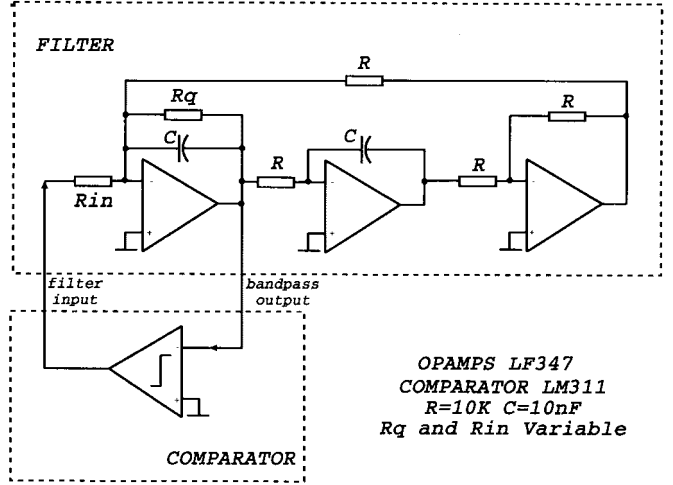


Fig. 6. Circuit schematic.

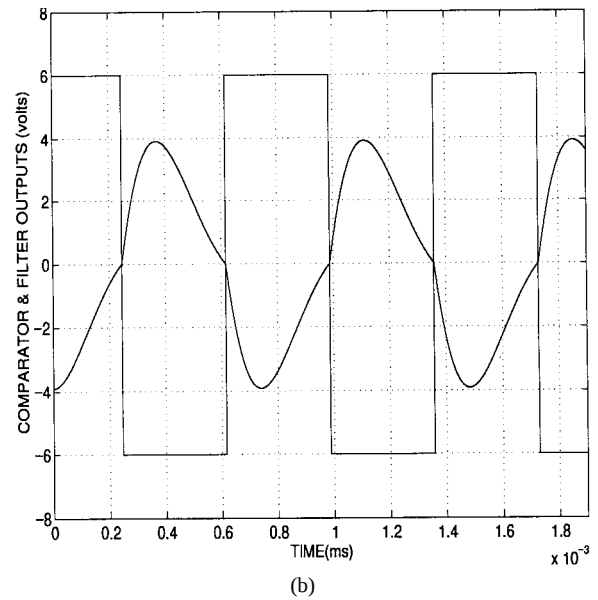
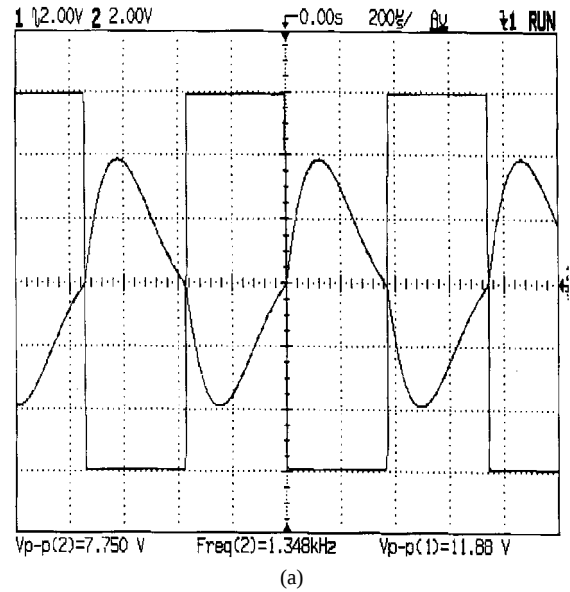
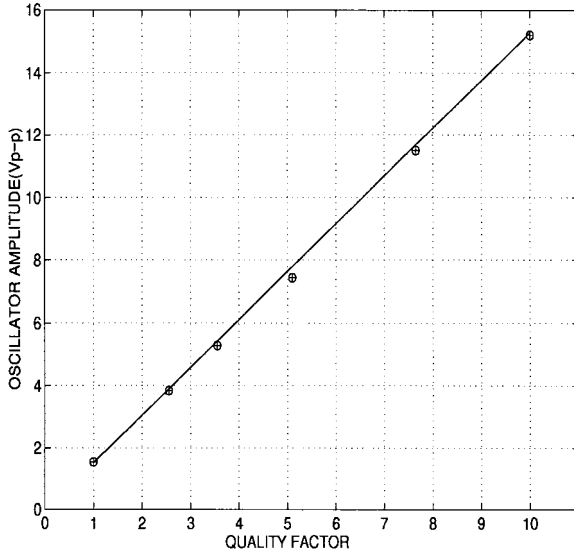


Fig. 7. Filter and comparator outputs: (a) measured and (b) computed.

Fig. 8. Measured ($\oplus$) and predicted (—) amplitude of oscillation versus Q .

and m is any integer. The peak of the oscillatory waveform is obtained by finding the maximum of $y_{ss}(mT + \tau)$ in the time interval $0 < \tau < T/2$, and is an exercise in calculus. We will find this peak for the special case when the filter quality factor is high, using (14) and (17):

$$y_{ss, \text{peak}} \cong A \cong \frac{1}{1 - \exp\left(-\frac{\pi}{\sqrt{4Q^2 - 1}}\right)} \cong \frac{1}{1 - \left(1 - \frac{\pi}{\sqrt{4Q^2 - 1}}\right)}, \quad Q \gg 1. \quad (18)$$

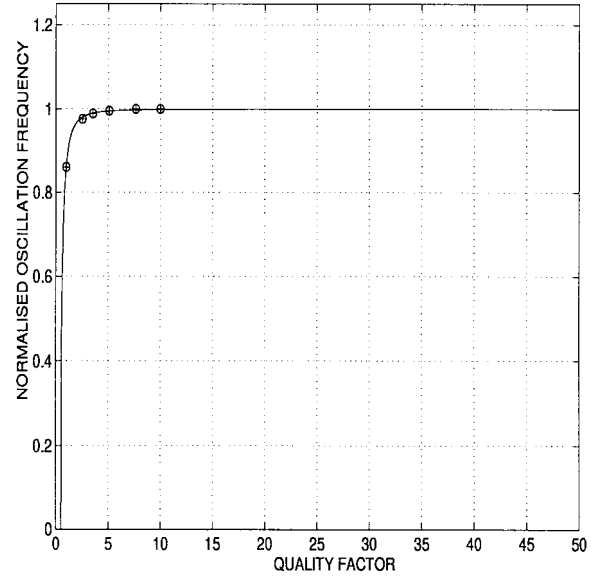
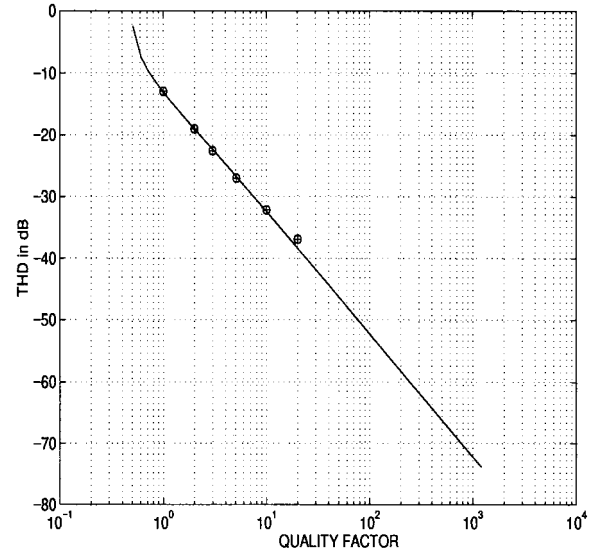
Thus,

$$y_{ss, \text{peak}} \cong A \cong \frac{2Q}{\pi}, \quad Q \gg 1. \quad (19)$$

Note that, all along, we have assumed the difference in the clipping levels of the comparator to be unity. In the more general case, the output will be directly proportional to the difference in clipping levels of the comparator. This proportionality will show itself as a multiplicative constant in (17). If the levels of the comparator are L_1 and L_2 ($L_1 > L_2$), then the analysis just presented can still be applied. But now, the step response $s(t)$ has to be calculated with nonzero initial conditions. After going through the analysis, it can be shown that the results (14)–(16) still hold, and that the multiplicative constant that must be inserted in (17) is $L_1 - L_2$. This comes as no surprise, because the bandpass filter rejects the dc component of its input which is nonzero if $L_1 + L_2$ is not zero.

B. Startup Transient

We now consider the nature of startup dynamics assuming an initially relaxed network. For this analysis, we will focus on the peak of the oscillator output within each half-period. The sequence of peaks can be considered to be a discrete

Fig. 9. Measured ($\oplus$) and predicted (—) frequency of oscillation versus Q .Fig. 10. Measured ($\oplus$) and predicted (—) THD versus Q .

time sequence, and successive peaks can be shown to occur at a time intervals of $T/2$ seconds. We will denote by a_n the peak of the absolute value of the output in the time interval $nt_1 < t < (n+1)t_1$. From Fig. 4, (3), (12), and (13), it is evident that we can write

$$\begin{aligned} a_n &= a_0 + \exp\left(-\frac{\omega_0 T}{4Q}\right)a_0 + \cdots + \exp\left(-\frac{\omega_0 nT}{4Q}\right)a_0 \\ &= a_0 \sum_{k=0}^n \exp\left(-\frac{k\omega_0 T}{4Q}\right). \end{aligned} \quad (20)$$

Since

$$\sum_{k=0}^n x^k = \frac{x^{n+1} - 1}{x - 1} \quad (21)$$

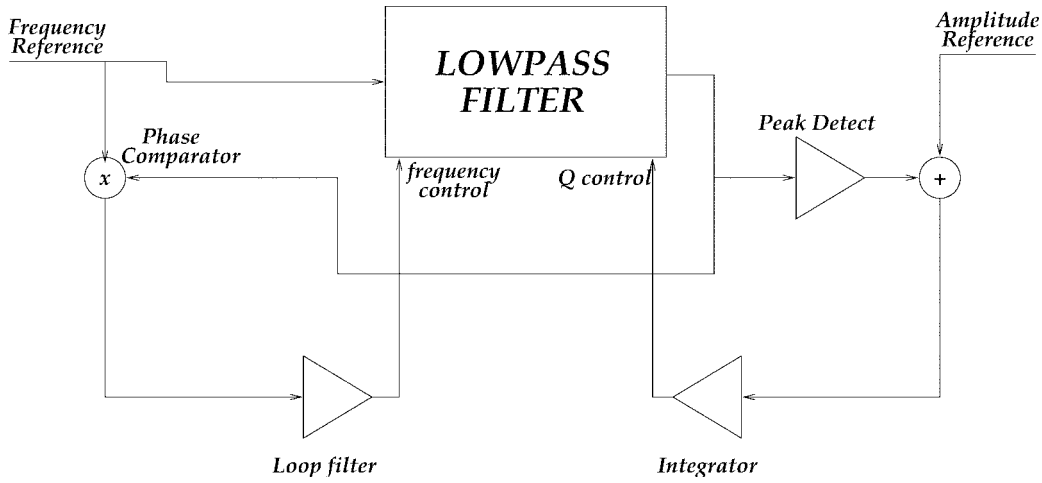


Fig. 11. A conventional vector lock loop (VLL).

we obtain

$$a_n = a_0 \left(\frac{a^{n+1} - 1}{a - 1} \right) \quad (22)$$

where

$$a = \exp \left(-\frac{\omega_0 T}{4Q} \right). \quad (23)$$

Note that $a < 1$ and the steady-state amplitude, obtained by setting $n = \infty$ in (22), is $a_0/(1 - a)$. The time taken for the output to reach 90% of its steady-state amplitude is $T/2(\log_a 0.1 - 1)$. This is obtained by putting $a_n = 0.9(a_0/(1 - a))$ in (22), solving for n , and calculating the required time as $nt_1 = n(T/2)$. As expected, as $a \rightarrow 1$ (equivalent to saying that $Q \rightarrow \infty$), the time taken to reach steady state tends to ∞ .

C. Harmonic Distortion Analysis

Based on the detailed analysis of the oscillator steady-state presented, the steady state response of the filter-comparator system can be obtained by considering an open-loop system in which the filter is driven by a periodic square wave, of frequency ω_{osc} . Expressing this square wave as a Fourier series, and calculating the attenuation offered to each Fourier component by the filter, it is straightforward [8] to calculate the spectrum at the output of the filter, and from this the total harmonic distortion (THD). This result can serve as a useful bound for THD when the comparator (amplifier) has finite gain. In that case, we assume that the frequency changes only very little from that predicted by the above analysis. So for the THD estimate, we say that the system is equivalent to a filter driven by an imperfect square wave with finite rise time, and thus with attenuated harmonics in comparison to a perfect square wave. It is thus reasonable to assume that THD in that case is bounded from above by that in the comparator case. This has been verified by simulations.

D. Experimental Results

We now present experimental results obtained with a bread-boarded prototype of the filter-comparator system. The circuit

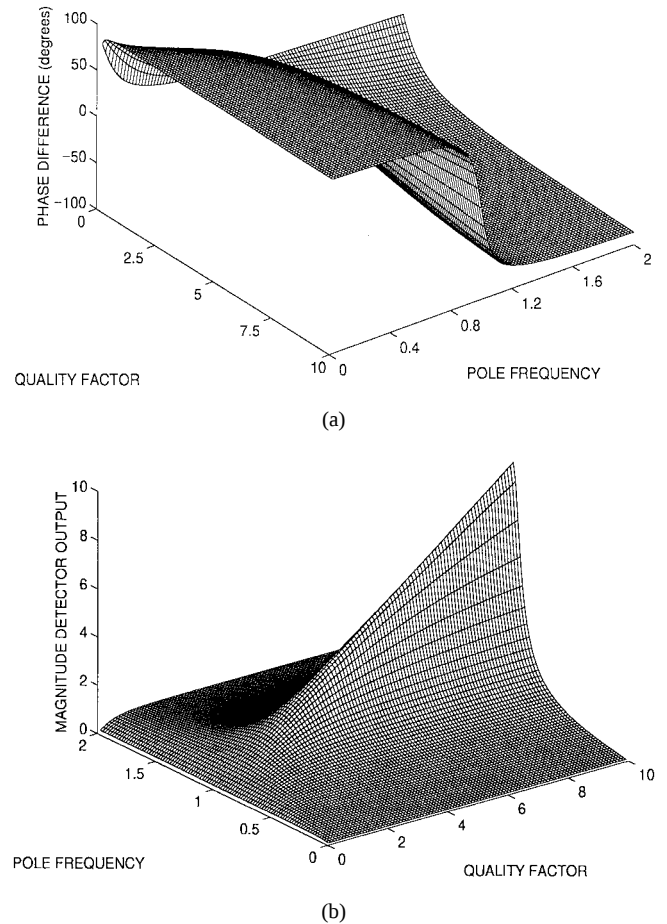
Fig. 12. (a) ϕ and (b) M surfaces for a conventional VLL (frequency normalized to reference).

diagram of the system is shown in Fig. 6. The filter section is a second-order op-amp RC filter, with pole frequencies in the low kilohertz range. The comparator used was LM311(National Semiconductor). As the quality factor was changed by varying the damping resistor of the biquad, the amplitude and frequency of oscillation changed in extremely good agreement with theoretical predictions.

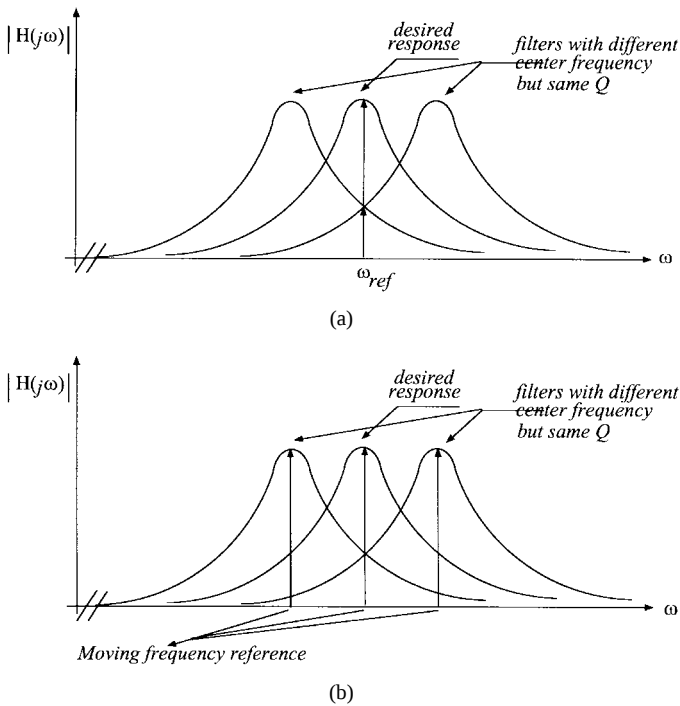


Fig. 13. Comparison of conventional and proposed technique.

In Fig. 7, we show the predicted and observed waveforms of the comparator and the filter when Q is one.

The measured and predicted amplitude of the output is shown in Fig. 8. Note that as the quality factor increases, the amplitude increases as predicted by (17). The frequency and total harmonic distortion (THD) as a function of filter quality factor are shown in Figs. 9 and 10, respectively. Notice that as the filter gets more selective, the harmonics of the output are attenuated to a greater degree, resulting in a lesser THD. From the figures, it is clear that experimental results agree very well with predictions.

III. APPLICATIONS TO FILTER TUNING

A long-standing problem in filter design has been to tune a filter to a desired response in the face of variations in temperature and other environmental factors, tolerances, and aging. Since tuning (even manual) of a high-order filter is complex, integrated tuning schemes have generally relied on manipulating the response of basic filter building blocks like biquadratic sections. For a tutorial review of filter tuning, the reader is referred to [9].

The tuning strategy can be indirect [10] or direct [11]. In either case, the filter to be tuned is a voltage (or current) controlled filter, that is, a filter whose parameters are “programmable” by a set of control voltages (or currents). For a second-order section, the parameters of greatest interest are the pole frequency and the pole quality factor. Hence these two parameters need to be tuned. Implementation of both frequency and Q loops is imperative in any high-frequency and/or high- Q filter design.

The general block diagram of a vector lock loop (VLL) based on a voltage controlled filter (VCF) is shown in Fig. 11 [12], [13]. This scheme is chosen in order to appropriately

introduce our proposed scheme in the sequel. The transfer function of the filter is

$$H(s) = \frac{s}{\frac{s^2}{\omega_0^2} + \frac{s}{\omega_0 Q} + 1}. \quad (24)$$

The variable of interest in the frequency control loop is ϕ , the phase difference between the reference and the output, while in the Q -lock loop, it is M , the magnitude of the output at the pole frequency

$$\phi(\omega_0, Q) = \frac{\pi}{2} - \arctan \left(\frac{\frac{\omega_{ref}}{Q\omega_0}}{1 - \frac{\omega_{ref}^2}{\omega_0^2}} \right) \quad (25)$$

$$M(\omega_0, Q) = \frac{\frac{\omega_{ref}}{\omega_0}}{\sqrt{\left(1 - \frac{\omega_{ref}^2}{\omega_0^2}\right)^2 + \left(\frac{\omega_{ref}^2}{\omega_0^2 Q^2}\right)}}. \quad (26)$$

The above equations show the coupled nature of the phase and magnitude measurements. To make the coupling effects even more explicit, the phase and magnitude detector output surfaces are drawn in Fig. 12.

We will now point out the problem with interloop coupling in a conventional VLL which uses a second-order filter. For this argument, the reader is referred to Fig. 13.

Fig. 13(a) shows the situation with the conventional vector locked loop. Assume that, to begin with, the relative shape of the response is very close to the ideal, while the center frequency deviates significantly from the desired value. For purposes of argument, assume that frequency and Q tuning is done sequentially. The magnitude detector will have an output which is very low, and this would cause the Q -loop to increase the filter Q , although there is only a frequency error in the system. Now, however, when the frequency loop converges to the desired value, the quality factor will be in error, and the magnitude loop now needs more time to get back to the right value. Notice that if the desired quality factor is large, then even a small error in pole frequency could result in the magnitude detector sensing a very low output. Thus the problems with locking tend to get compounded with increasing filter selectivity. In traditional schemes, these problems are taken care of by making the Q -loop much slower than the frequency loop, so as to make the loops quasi-independent. Note that, ideally, we would want

$$\frac{\partial \phi(\omega_0, Q)}{\partial Q} = 0 \quad (27)$$

$$\frac{\partial M(\omega_0, Q)}{\partial \omega_0} = 0. \quad (28)$$

From Fig. 13(a), it is obvious that all the problems with the conventional design could be avoided if we were somehow able to “move” the reference around, so that we can always sense the peak gain of the filter, no matter at what frequency it occurs. This situation is illustrated in Fig. 13(b). Now, the

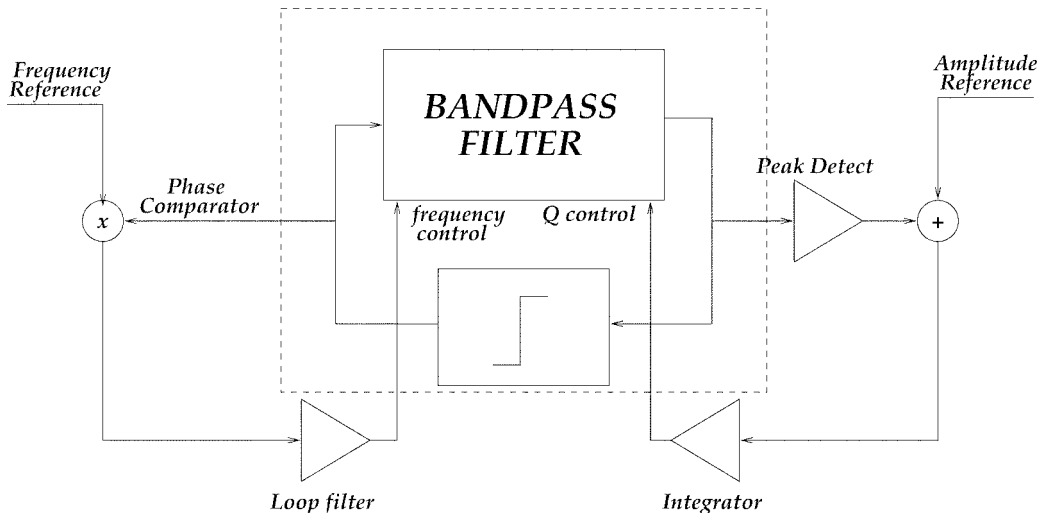


Fig. 14. Proposed VLL.

magnitude detector output is constant regardless of filter center frequency, and a function of quality factor only. To generate a “reference frequency” which is always equal to the filter pole frequency, one can excite the filter and pass its output through a limiter to obtain a constant amplitude. This is precisely what the system of Fig. 3 does. From (17), it is apparent that the amplitude of oscillation is now a single-valued function of filter quality factor only, and is completely independent of pole frequency.

The entire vector lock loop is shown in Fig. 14. The pole frequency of the filter is set by locking the oscillation frequency to the reference using a phase-lock loop. The quality factor is set by measuring output magnitude. From (15), we see that the oscillation frequency is an extremely weak function of Q . As a numerical example, the difference between the oscillation frequency when Q changes from 5 to 20 (a change in Q of 300%) is just 2%. Thus, we can conclude that the oscillation frequency is essentially independent of Q for reasonably high values of Q . The frequency and amplitude of oscillation as a function of normalized pole frequency and quality factor are shown in Fig. 15. The independence of magnitude and frequency measurements is apparent from these two surface plots. We now discuss how the scheme just presented is different from classical VCO methods discussed in the literature. The classical methods also use the PLL principle and amplitude stabilization, but they focus on single integrators, as opposed to the biquadratic section in the proposed scheme. The VCO's implemented in the traditional schemes limit amplitude of oscillation using nonlinear methods, but assume that the frequency of oscillation remains that of the resonator, which is incorrect. Although the oscillation frequency will be close to the frequency of the resonator, it will nevertheless be dependent on the nature of the nonlinearity of the amplitude stabilizing element [4], [6]. In indirect tuning methods, this makes tight tracking between master and slave difficult. Our scheme operates the filter within its limits of linearity, and can be used around a resonator with any Q greater than 0.5. An alternate solution to keep the filter operating in a linear mode is to use an AGC circuit instead of the comparator

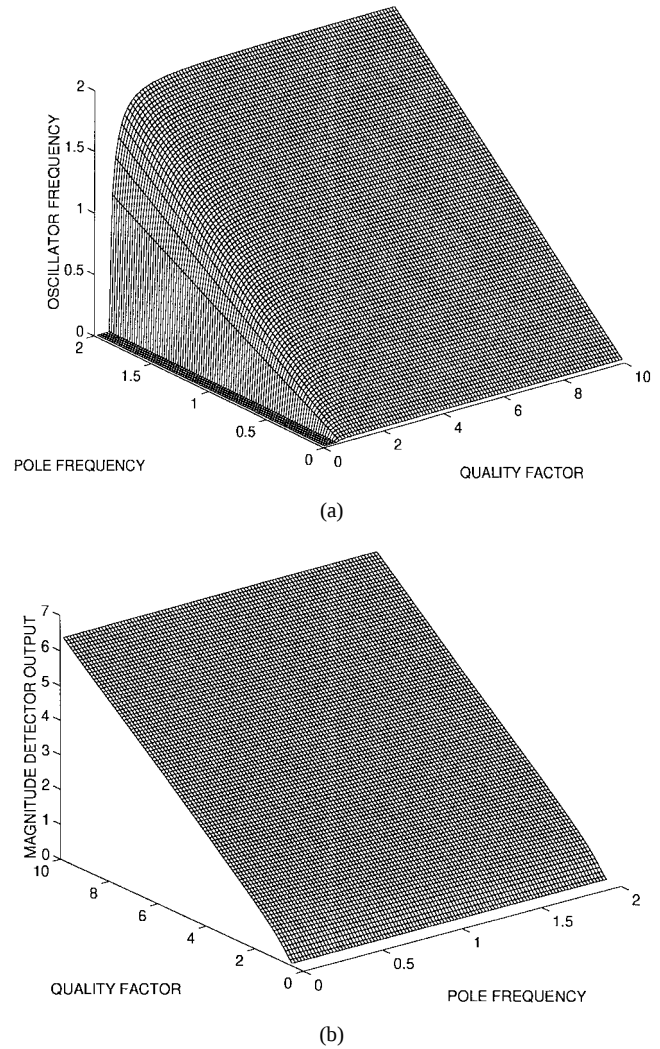


Fig. 15. (a) Frequency and (b) amplitude surfaces for proposed VLL (frequency normalized to desired pole frequency).

in Fig. 3. Tuning of infinite Q filters by this method has been proposed in [14]–[16]. Note that even in this case, the amplitude and frequency loops are independent [14]. This is

more complicated to implement than the comparator method, and does not offer us the convenience of a square wave output (which is readily available in the comparator case). We now summarize the advantages of the VLL just presented.

- 1) The pole frequency can be tuned with absolutely no error in spite of offsets in the frequency control loop because the system utilizes the PLL principle, in which phase errors do not result frequency errors.
- 2) The reference can be a square wave, unlike in the VCF case, which demands a reference signal with low harmonic content.
- 3) The filter operates in a linear fashion, and the oscillation frequency of the entire system tracks the pole frequency of the filter with variations in ambient conditions and other environmental factors.
- 4) The amplitude and frequency loops are independent.
- 5) This can be used in direct tuning schemes, because the filter to be used can be tuned directly in contrast to conventional VCO schemes which tune individual integrators.

Thus, this loop is a marriage of the VCF and the PLL schemes, combining the advantages of both in the same method, and getting rid of the disadvantages of either methods. The loop has the same circuit complexity as any other VLL scheme.

A. Experimental Results

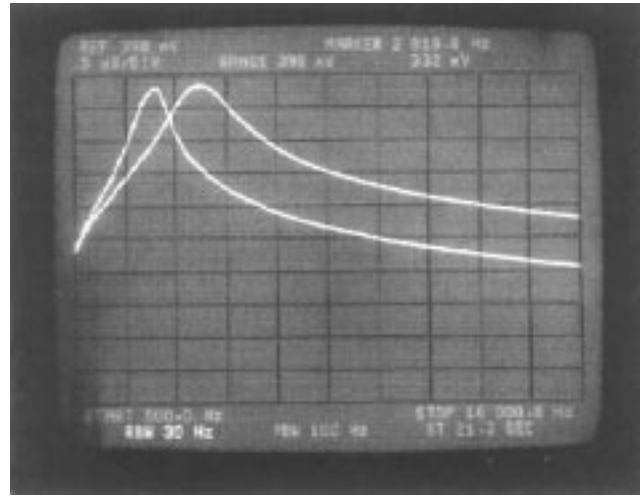
A low-frequency version of the proposed (VLL) was breadboarded. The master-slave system was realized by using MOS transistor arrays. The filter topology was a second-order filter of the Tow-Thomas kind, with tunable pole frequency and quality factor. The comparator used was an LM311 (National Semiconductor Corp.). The pole frequency and quality factor of the filter were observed to adjust to the reference frequency and the dc voltage reference to the magnitude locked loop. Setting these quantities, filter tuning could be accomplished for reference frequencies of 1.4–2.7 kHz, and Q values from 1–6. No special steps were adopted for stabilizing the loops, and we encountered no problems with stability of either loop. The limited capture and lock ranges of the PLL were due to the fact that no attempt was made to optimize the design, which was done just to check functionality of the VLL. Fig. 16 shows the functionality of the frequency and Q loops.

IV. APPLICATIONS TO SWITCHED MODE RESONANT OSCILLATORS

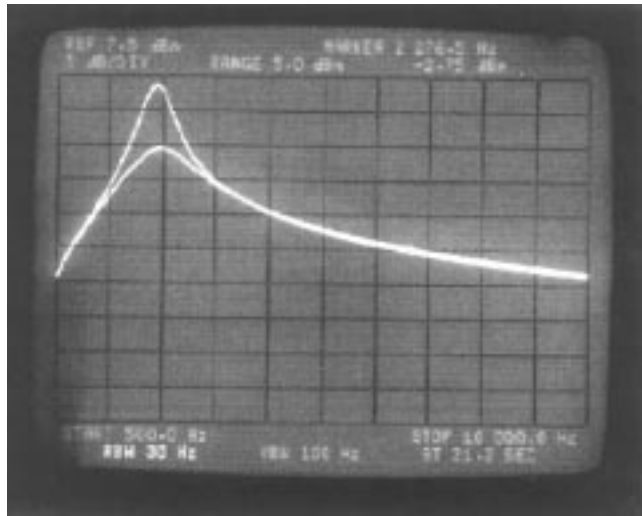
Switched mode resonant oscillators form a useful class of systems, finding application in dc to ac conversion. The schematic of a switching mode resonant dc-to-ac inverter is shown in Fig. 17. If the voltage across the series LCR network is written as V , note that the current can be written as

$$I(s) = \frac{sC}{s^2LC + sCR + 1} V(s). \quad (29)$$

The polarity of the voltage source switched across the circuit is dependent on the zero crossings of the current. Notice that the current is a bandpass-filtered version of the voltage, and



(a)



(b)

Fig. 16. Functionality testing of the proposed VLL: (a) varying center frequency by varying frequency reference and (b) varying quality factor by varying voltage reference.

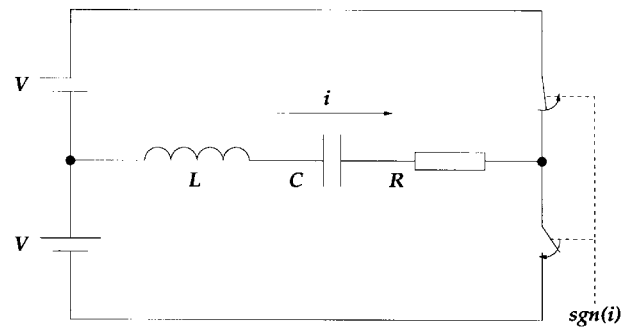


Fig. 17. A switching mode resonant dc-to-ac converter.

hence the system is exactly equivalent to the system shown in Fig. 2. Hence, the analysis we have presented holds in its entirety. In this case

$$\omega_o = \frac{1}{\sqrt{LC}} \quad (30)$$

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}}. \quad (31)$$

The behavior of this system has been investigated by numerical simulation in [17] and [18]. The conclusion reached in the references is the same—the minimum Q required for oscillation is $1/2$. The references cited above also investigate the behavior of this oscillator at the edge of oscillation, that is, at a Q of around the above critical value.

V. CONCLUSIONS

In this paper, we have presented an analytical technique for the solution of a class of sinusoidal oscillators. A vector lock loop, based on this class, has been proposed. The individual loops of this VLL are uncoupled. This scheme combines the best of both the VCF and VCO schemes. We also discussed the applications of our solution to resonant switched mode inverters.

APPENDIX

For this analysis, the reader is referred to Fig. 5. We use $T = 2t_1$ from (13), and denote by k the even integer $2m$. Then, from (12), we get

$$y(kt_1 + \tau) = \sum_{n=0}^k (-1)^n s(kt_1 + \tau - nt_1), \quad 0 < \tau < t_1. \quad (32)$$

Using a change of variables, and keeping in mind that k is even, for $\tau < t_1$, (29) can be written as follows:

$$y(kt_1 + \tau) = \sum_{n=0}^k (-1)^n s(\tau + nt_1). \quad (33)$$

In order to avoid unwieldy expressions, and in preparation for the development that follows, we use the notation

$$\omega_{osc} = \omega_0 \sqrt{1 - \frac{1}{4Q^2}}. \quad (34)$$

Equation (3) for $t = \tau + nt_1$ becomes, using (6) and (31),

$$s(\tau + nt_1) = \frac{1}{\sqrt{1 - \frac{1}{4Q^2}}} \exp\left(-\frac{n\omega_0 t_1}{2Q}\right) \exp\left(-\frac{\omega_0 \tau}{2Q}\right) \cdot \sin(\omega_{osc} \tau + n\pi). \quad (35)$$

Using (32) in (30), and noting that $(-1)^n \sin(x + n\pi) = \sin(x)$, we get

$$y(\tau + kt_1) = \sum_{n=0}^k \frac{1}{\sqrt{1 - \frac{1}{4Q^2}}} \exp\left(-\frac{n\omega_0 t_1}{2Q}\right) \exp\left(-\frac{\omega_0 \tau}{2Q}\right) \cdot \sin(\omega_{osc} \tau), \quad 0 < \tau < t_1 \quad (36)$$

or, using (6),

$$y(kt_1 + \tau) = \frac{\exp\left(-\frac{\omega_0 \tau}{2Q}\right) \sin(\omega_{osc} \tau)}{\sqrt{1 - \frac{1}{4Q^2}}} \sum_{n=0}^k \exp\left(-\frac{n\pi}{\sqrt{4Q^2 - 1}}\right), \quad 0 < \tau < t_1. \quad (37)$$

To calculate the steady-state response, we allow k to increase. In the limit, replacing the sum by an infinite sum, and using

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1 \quad (38)$$

the right hand side of (34) becomes

$$\frac{\exp\left(-\frac{\omega_0 \tau}{2Q}\right) \sin(\omega_{osc} \tau)}{\left[1 - \exp\left(-\frac{\pi}{\sqrt{4Q^2 - 1}}\right)\right] \sqrt{1 - \frac{1}{4Q^2}}}, \quad 0 < \tau < t_1. \quad (39)$$

It is obvious from Fig. 4 that for $t_1 < \tau < 2t_1$, the steady-state response shape is the same as in the interval $0 < \tau < t_1$, except for a sign inversion. Thus, (14)–(17) in Section II hold, where T is as in (13).

ACKNOWLEDGMENT

The authors sincerely thank K. Nagendra for the bread-board implementation of the VLL. They also wish to thank the anonymous reviewers for their suggestions.

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