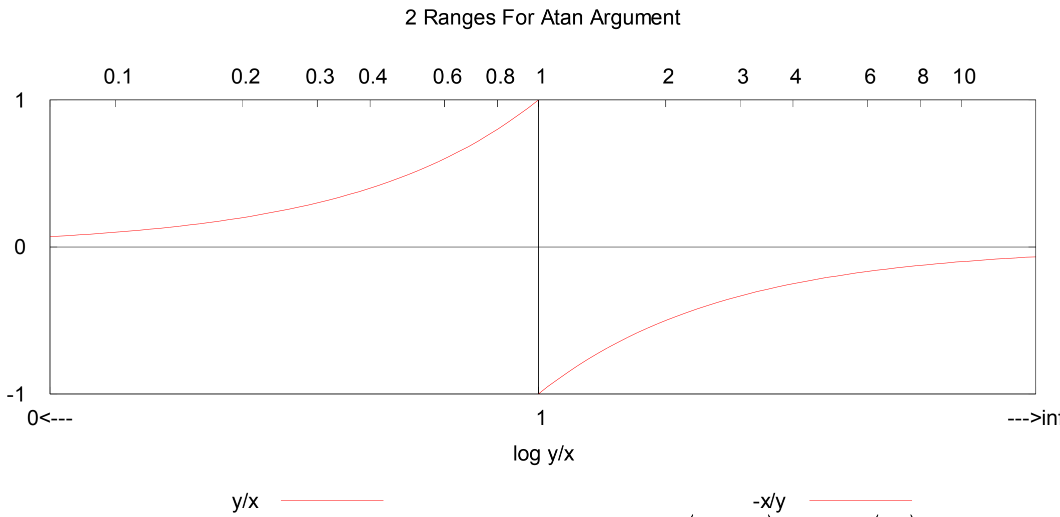
Derivation Of Efficient Arctan Algorithm

by Satin Hinge



An attempt at evaluating *atan* would be:

$$atan\left(\frac{y}{x}\le 1\right)=atan\left(\frac{y}{x}\right)$$

$$atan\left(\frac{y}{x}>1\right)=\frac{\pi}{2}-atan\left(\frac{x}{y}\right)=\frac{\pi}{2}+atan\left(-\frac{x}{y}\right)$$

Which suggests $atan(x1\le s\le x2)=p+atan(-q)$ might be a formula that would allow us to create more ranges.

Since -*q* is just another constant, we'll absorb it into a new constant t to give:

$$atan(s)=p+atan(t)$$

Utilizing $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ we take the *tan* of both sides.

$$\tan(atan(s))=\tan(p+atan(t))\rightarrow$$

$$s=\frac{\tan p+t}{1-t\tan p}$$

$$t=\frac{s-\tan p}{1+s\tan p}$$

To get rid of the *tan* p term, let us replace the constant in our orginal formula by an equivalent constant k = *tan* p to give.

$$\boxed{t=\frac{s-k}{1+sk}}$$

And since p = *atan* k, this translates our original formula into:

$$\boxed{atan(s) = atan(k) + atan(t)}$$

Notice if we let $k \to \infty$

$$\lim_{k \to \infty} \left[atan(k) + atan\left(\frac{s-k}{1+sk} \right) \right] = \frac{\pi}{2} + atan\left(-\frac{1}{s} \right)$$

and if we let k = 0

$$\lim_{k \to 0} \left[atan(k) + atan\left(\frac{s-k}{1+sk} \right) \right] = atan(s)$$

Which means we've derived a more generalized formula.

So far we've got 2 ranges:

$$\begin{matrix} 0 \le s \le 1 & k = 0 \\ 1 \le s \le \infty & k = \infty \end{matrix} \quad -1 \le t \le 1$$

We'd like to insert more ranges:

$$\begin{matrix} 0 \le s \le x_1 & k = 0 \\ x_1 \le s \le x_2 & k = k_1 \\ x_2 \le s \le x_3 & k = k_2 \\ \vdots & \vdots \\ x_{n-1} \le s \le x_n & k = k_{n-1} \\ x_n \le s \le \infty & k = \infty \end{matrix} \quad -r \le t \le r \quad \text{and solve for the x's and k's}$$

We know from the graph above that t jumps from positive to negative at each gap. We therefore want to constrain the value of t at each gap to the magnitude r.

We can express these constraints in the form of equations:

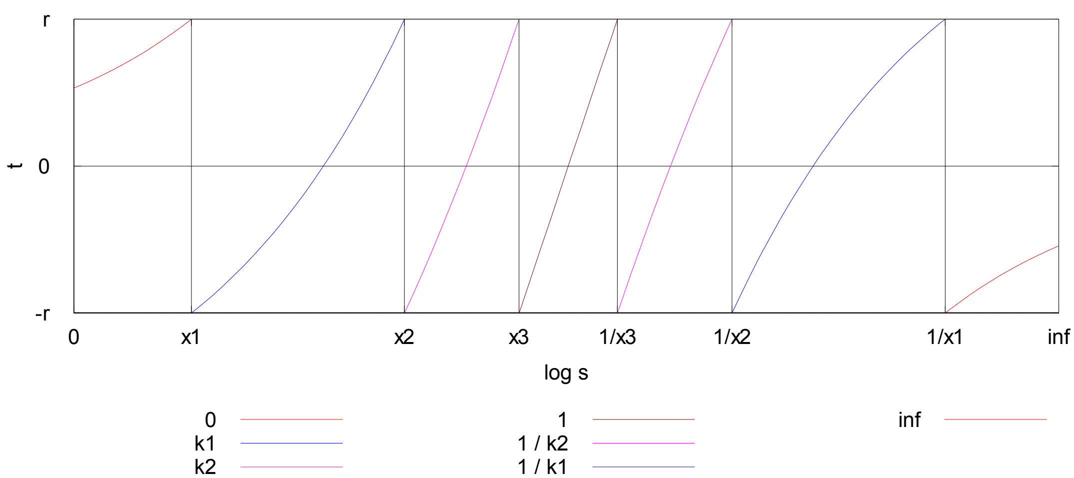
$$
\begin{aligned}
t_{s \to x_1 -} &= r \\
t_{s \to x_1 +} &= -r \\
t_{s \to x_2 -} &= r \\
t_{s \to x_2 +} &= -r \\
&\vdots \\
t_{s \to x_n -} &= r \\
t_{s \to x_n +} &= -r
\end{aligned}
$$

Yields a system of $2\boldsymbol{n}$ non-linear equations in $2\boldsymbol{n}$ variables:

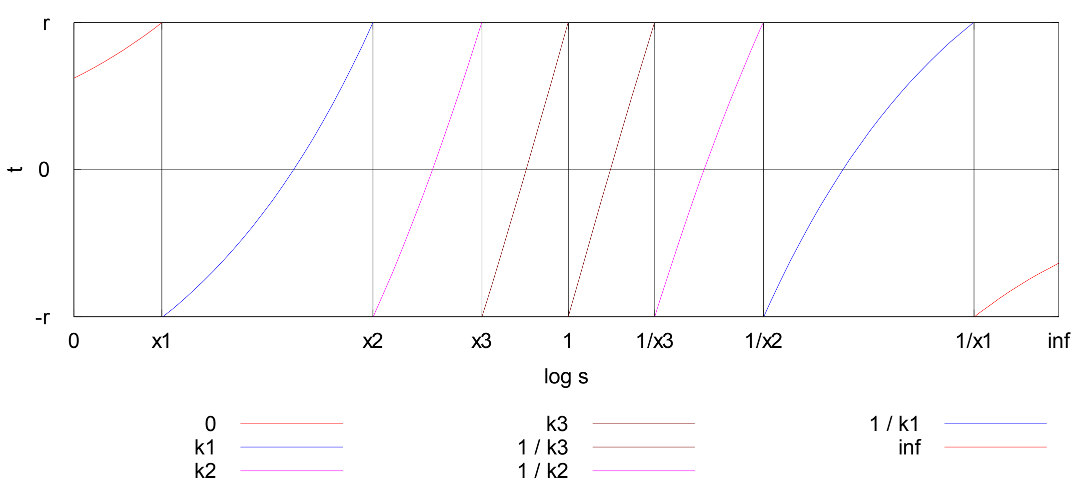
$$
\begin{aligned}
x_1 &= r \\
\frac{x_1 - k_1}{1 + x_1(k_1)} &= -r \\
\frac{x_2 - k_1}{1 + x_2(k_1)} &= r \\
\frac{x_2 - k_2}{1 + x_2(k_2)} &= -r \\
&\vdots \\
\frac{x_n - k_{n-1}}{1 + x_n(k_{n-1})} &= r \\
-\frac{1}{x_n} &= -r
\end{aligned}
$$

Or so it appears. After solving a few systems, a pattern begins to emerge:

Odd Number Of Ranges



Even Number Of Ranges



Apparently by making such symmetrical demands on the equations, the equations have responded in kind.

So instead of p ranges requiring 2(p-1) variables, they only require p-1 variables. And if we count the fact x1 = r, that's p-2 variables.

Because of this, exact solutions exist for up to 7 partitions:

2 Ranges

$$x_1=1$$
$$r=1$$

3 Ranges

$$x_1=\sqrt{(2)}-1$$
$$k_1=1$$
$$x_2=\sqrt{(2)}+1$$
$$r=\sqrt{(2)}-1$$

4 Ranges

$$x_1=2-\sqrt{(3)}$$
$$k_1=\frac{1}{\sqrt{(3)}}$$
$$x_2=1$$
$$k_2=\sqrt{(3)}$$
$$x_3=2+\sqrt{(3)}$$
$$r=2-\sqrt{(3)}$$

5 Ranges

$$x_1=\sqrt{4+2\sqrt{2}}-\sqrt{2}-1$$
$$k_1=\sqrt{2}-1$$
$$x_2=\sqrt{4-2\sqrt{2}}-\sqrt{2}+1$$
$$k_2=1$$
$$x_3=\sqrt{4-2\sqrt{2}}+\sqrt{2}-1$$
$$k_3=\sqrt{2}+1$$
$$x_4=\sqrt{4+2\sqrt{2}}+\sqrt{2}+1$$
$$r=\sqrt{4+2\sqrt{2}}-\sqrt{2}-1$$

6 Ranges

$$x_1=\sqrt{5}+1-\sqrt{5+2\sqrt{5}}$$
$$k_1=\frac{1}{\sqrt{5}}\sqrt{5-2\sqrt{5}}$$
$$x_2=\sqrt{5}-1-\sqrt{5-2\sqrt{5}}$$
$$k_2=\sqrt{5-2\sqrt{5}}$$
$$x_3=1$$
$$k_3=\frac{1}{\sqrt{5}}\sqrt{5+2\sqrt{5}}$$
$$x_4=\sqrt{5}-1+\sqrt{5-2\sqrt{5}}$$
$$k_4=\sqrt{5+2\sqrt{5}}$$
$$x_5=\sqrt{5}+1+\sqrt{5+2\sqrt{5}}$$
$$r=\sqrt{5}+1-\sqrt{5+2\sqrt{5}}$$

7 Ranges

$$x_1=\sqrt{6}-2-\sqrt{5-2\sqrt{6}}$$
$$k_1=2-\sqrt{3}$$
$$x_2=\sqrt{2}-1$$
$$k_2=\frac{1}{\sqrt{3}}$$
$$x_3=\sqrt{6}-2+\sqrt{5-2\sqrt{6}}$$
$$k_3=1$$
$$x_4=\sqrt{6}+2-\sqrt{5+2\sqrt{6}}$$
$$k_4=\sqrt{3}$$
$$x_5=\sqrt{2}+1$$
$$k_5=2+\sqrt{3}$$
$$x_6=\sqrt{6}+2+\sqrt{5+2\sqrt{6}}$$
$$r=\sqrt{6}-2-\sqrt{5-2\sqrt{6}}$$